Analysis of Optimal Composite Feedback-Feedforward Control

R. H. LUECKE and M. L. MCGUIRE
University of Oklahoma, Norman, Oklahoma

We have developed analytic design methods for combination feedback-feedforward control systems and have evaluated systems yielding optimal performance while subject to constraints commonly encountered in the chemical industry. Using the mathematical techniques pioneered by Wiener for the solution of the design equations, we have based the optimization on minimization of the mean square output of a system subject to a random disturbance. Side conditions for the constraint on mean square control effort, signal-to-noise ratio in the feedback system, and minimization of error output caused by misidentification of plant parameters were found to be necessary to give physically realizable and meaningful designs. The analytic design methods are useful for analysis of control system performance and capabilities under a variety of constraints, but the optimal designs are marginally superior to ideal or invariant feedforward controllers coupled with tuned proportional feedback controls. The principal improvement in the optimal design is in conservation of control effort when compromises in system performance are necessary because of this restriction.

The art and science of feedforward controller design in the process industries began about fifteen years ago with the study of cascade control. Although conventional feedback methods still dominate the field, interest in feedforward has increased steadily as process control engineers have realized that the existence of a continuous monitor of disturbances made feedforward control particularly suitable for many chemical process problems (*3*). While some investigators have implied that feedforward alone may be sufficient for control purposes, most have concluded that a combination of feedback and feedforward is needed in the majority of cases. Undoubtedly feedforward control will be "encountered more frequently in the future as an essential aspect of composite control" (*2*). The objective of the present work is to investigate optimal composite control systems for chemical processes.

Most of the previous consideration of feedforward control has not been concerned with explicit optimization methods. The reason is that if the exact mathematical model is known, a controller can be constructed which is the mirror image of the plant so that when the controller output is added to that of the plant, disturbance attenuation is perfect. Harris and Schechter (*5*) and Bollinger and Lamb (*1*), in their work on chemical reactors, specified mirror images of linear approximations to the system describing functions for the feedforward portion of their controllers. The feedback section of these control systems, chosen in both cases by cut-and-try procedures, compensated only for the plant nonlinearities and for inaccuracies of analog computer programming. Haskins and Sliepcevich (*6*), in their study of the invariance principle, used nominal feedforward controllers with no primary feedback but showed how compensation of analytic nonlinearities may also be programmed in the feedforward section of the controller.

In feedforward control of distillation columns, secondary variables are used as indications of output changes and are made invariant by mirror image control (*12, 14*). Feedback usually consists of manual adjustment of drift although several authors have advocated development of more sophisticated feedback techniques (*13, 14*).

Analytic design methods based on the calculus of variations approach for *feedback* control systems were investigated by Newton, Gould, and Kaiser (*16*). The optimization method was based on the minimization of the mean square error between an ideal reference and the actual output in the presence of constraints, and in particular, constraints on control effort. The equations defining necessary conditions for the minimum were solved by using an advanced complex variable theory which, as demonstrated by Wiener (*20*), yields an explicit solution for the control function.

In the present work, the same fundamental mathematical technique is employed; the main difference is in the nature of the constraints which are utilized and the fact that two control functions are computed, one relating to feedback alone and another overall control function for both feedback and feedforward.

Chang (*4*) noted that continuous measurement of the reference signal and of the plant output was sufficient to specify two control functions separately, one an open loop function compensating for reference set point changes and another a feedback control function compensating for results of an external or load disturbance. More generally, Horowitz (*8*) showed that as many control functions can be specified as there are independent measurements of system variables. In the present work, the ideal reference is assumed to be identically zero so that Chang's open loop reference point control becomes trivial.

The use of Wiener's methods for optimal control design has been extended by others (*9, 10, 15, 17*) but only single control functions are defined for each output. Constraints have been discussed in general but the cases studied use constraints which often yield controllers which are not physically realizable under conditions of measurement and model accuracy existent in chemical process systems.

STATEMENT OF PROBLEM

We assume that the objective of the control design and operation is to attenuate the output response of chemical process systems which are activated by random disturbances. In addition, we assume that the process system or plant can be adequately described by ordinary linear differential equations with constant coefficients; in fact, we will give specific consideration to controllers which cause the plant to operate in the linear range for a large fraction of the time. Only single variable plants are treated, that is, plants which have only one disturbance variable and one controlled output variable. The

R. H. Luecke is with Monsanto Comany, St. Louis, Missouri.

generalization to n disturbances is a trivial multiplication of feedforward controllers but the generalization to n outputs is far from trivial.

Under these conditions, the process dynamics can be described by the differential equation

$$\frac{d^n c}{dt^n} + a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} + \ldots + a_o c$$
$$= b_j \frac{d^i m}{dt^i} + \ldots + b_o m + g_k \frac{d^k d}{dt^k} + \ldots + g_o d \quad (1)$$

where the a's, b's and g's are constants, c is the output or controlled variable, m is the manipulative or controlling variable, and d is the load or disturbance variable.

It is most convenient to work with the Laplace transformation of this equation that results from assuming zero initial conditions. Transforming (1) into the Laplace domain yields

$$C(s) = P_D(s)\ D(s) + P_M(s)\ M(s) \quad (2)$$

where $P_D(s)$ and $P_M(s)$ are the plant transfer functions. If dead times are absent, these functions are rational functions in the Laplace transform variable s, while if dead times are present the rational functions are multiplied by an exponential factor in s. The quantities $C(s)$, $M(s)$ and $D(s)$ are the Laplace transformations of $c(t)$, $m(t)$ and $d(t)$, respectively.

The design objective is to define transfer functions for controllers, $Q_C(s)$ and $Q_D(s)$, so that a minimum of the absolute value of $C(s)$ will occur when the manipulative variable is defined as

$$M(s) = Q_D(s)D(s) - Q_C(s)C(s) \quad (3)$$

The function $Q_C(s)$ is the feedback transfer function since its control action is based on information feedback from the output while $Q_D(s)$ is the feedforward transfer function since its control action is based on information from the input.

It is clear that the minimum absolute value of system response occurs if the output variable can be made identically zero. Defining the manipulative variable as

$$M(s) = -\frac{P_D(s)}{P_M(s)} D(s) \quad (4)$$

and substitution of this expression for $M(s)$ in (1) results in the output variable c being identically zero. The relation (4) can be realized formally if the feedforward control function is defined as

$$Q_D(s) = -\frac{P_D(s)}{P_M(s)} \quad (5)$$

It would seem unnecessary to search for an optimal controller if a control function has been found which makes the output identically zero; clearly an optimal controller can do no better since the objective of the present design has been stated to attenuate the output. There are a number of reasons why the ideal solution (5) cannot be realized in practice. The optimal control design will result from a study of this constraints.

CONTROL WITH CONSTRAINTS ON CONTROL EFFORT

In any real process, a limit exists for the size of the manipulative variable, m. If, for example, the process is utilizing cooling water for temperature control, a disturbance may occur of sufficient magnitude so that the cooling flow demand of the controller (5) exceeds the flow capacity. In this case, the system output will not remain identically zero and there is no a priori reason to believe that the control law (5) is the optimal one. This problem may be expressed in terms of calculus of variations as follows: Given a system described by equation (1) or (2), find Q_D and Q_C of equation (3) such that the magnitude of the output c is minimized when the controlled system is activated by a random disturbance, D. At the same time, the system is subject to the constraint that a limit exists on the magnitude of the manipulative variable, that is,

$$|m| < \mathcal{L} \quad (6)$$

Explicit solution of this mathematical problem was achieved by Wiener (*20*) and details of the technique are available elsewhere (*11, 16*).

The mean square value of the variables will be taken as the measure of merit of the control system under consideration. Minimization of the mean square value of the output $\langle c^2 \rangle$ in the presence of a constraint on the control effort is achieved by minimizing a weighted sum of the quantities

$$F = \langle c^2 \rangle + \lambda^2 \langle m^2 \rangle \quad (7)$$

As the mean square value of available control effort increases, the achievable level of mean square output decreases. Thus $\langle c^2 \rangle$ is a monotonic function of $\langle m^2 \rangle$ and for every value of λ^2 for which F in (7) is minimized, there exist unique values of $\langle c^2 \rangle$ and $\langle m^2 \rangle$. The mean of the output and manipulative variables is zero so that the mean square value of each of these quantities is also its variance. Thus, for a given value of λ^2, minimization of F in (7) yields a unique minimum value of the variance of the output for a unique value of the variance of the manipulative variable.

The fraction of time that the square of a variable exceeds its variance depends on its distribution function. If the square root of the variance, that is, the standard deviation, of the manipulative variable in (7) is set equal to $\mathcal{L}$ in (6), then minimization of the output of system (1) subject to a gaussian input will be achieved with the manipulative variable obeying the inequality (6) about 68% of the time. If the standard deviation of m in (7) is made equal to $\mathcal{L}/2$, then (6) is valid about 95% of the time. Thus the condition (6) can be made to apply for the system operation for any arbitrary fraction of time by adjusting the value of the weighting factor in (7) until an acceptable level for $\langle m^2 \rangle$ is obtained.

This technique was applied to a process with simple first order transfer functions

$$C = \frac{K_D}{\alpha + s} D + \frac{K_M}{\alpha + s} M \quad (8)$$

where $\alpha = 2$ and scaling of variables is such that $K_D = K_M = 1$.

Calculation of the optimal control function requires not only the values of the plant transfer function but also statistical characterization of the random disturbance. We will assume here that the spectral density of the disturbance can be adequately represented as

$$\Phi_{DD}(s) = \frac{2\mu^2\sigma}{\sigma^2 - s^2} \quad (9)$$

This equation is the exact representation for the following processes: (1) constant magnitude square waves whose sign changes are a random function of time; (2) square waves with randomly changing signs and amplitudes; and (3) a gaussian noise produced by passing "white" noise through a first order filter. In general, random signals with identical statistical characteristics may

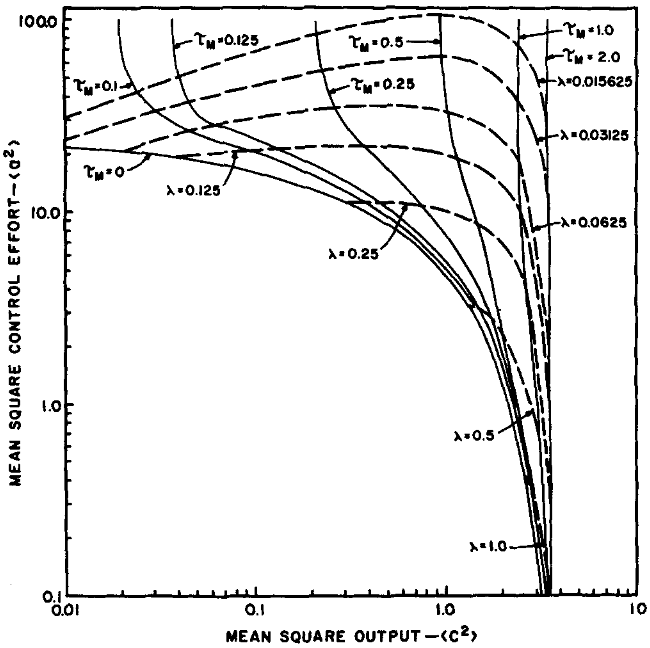

Fig. 1. Response characteristics of optimal feedforward control of first order system for various controller dead times, τ_M. Results are from minimization of sum $<c^2> + \lambda^2 <a^2>$.

have quite different time behavior *(10, 18)*.

The control effort can be constrained in several ways: (i) by the maximum level which it can attain [as given in (6)]; (ii) by some maximum rate at which it can change; (iii) by some maximum value of a linear combination of these and other operators on the manipulative variable. Thus let

$$A = P_A M \tag{10}$$

where P_A is some linear operator in the Laplace trans-from space and let the constraining condition be

$$|a(t)| < \mathcal{L} \tag{11}$$

where a (t) is the inverse transform of A. The function to be minimized becomes

$$F = <c^2> + \lambda^2 <a^2> \tag{12}$$

For the present, only the maximum value of the control effort is constrained as given by Equation (6) and hence P_A is taken to be equal to 1.

The digitally computed performance diagram for this model (Figure 1) shows the variation of mean square output $<c^2>$ as a function of mean square control effort $<a^2>$. The curve for dead time $\tau_M = 0$ applies to the present case. The output shown by this curve asymptotically approaches the uncontrolled value as the control effort approaches zero. The mean square control effort required to make the output approach zero also approaches a finite asymptote. The value of the asymptote is the control effort required by the ideal feedforward controller (5) and depends on the transfer functions P_D and P_M as well as on the magnitude of the disturbance. The shallow slope for small values of mean square output indicates that a high price in terms of output compensation is required for even a modest reduction in control effort. Alternatively, small constraints of control effort cause serious deterioration of optimal control performance.

There is a broad variation in control effort-output relationships as a function of the mean disturbance frequency. However much less variation occurs in the resulting parameters of the optimal controller. For the model and conditions under consideration, the control objective can be achieved without feedback (that is, $Q_C = 0$). The design equations yield a feedforward controller of the form

$$Q_D = -\frac{T_0 + T_1 s}{1 + \beta s} \tag{13}$$

where T_0', T_1 and β are constants *(11)*. The mean disturbance frequency does not enter into the computation of β and there is only modest effect on the value of T_0 and T_1 over a broad disturbance frequency range (Table 1). Therefore, for the remainder of these calculations a mean disturbance frequency of 1.5 will be assumed as compared to the system natural frequency of 2.0.

Several of the optimal controllers were studied on an analog computer and compared with the ideal feedforward controller of (5). Controller saturation in a chemical plant was simulated by electrical clipping of the controlling variable. System response to this nonlinearity cannot be analytically computed by methods heretofore considered. The disturbance was obtained by filtering a square wave having a mean random frequency twenty times higher than that required in the testing so that the resulting noise was approximately gaussian.

In Figure 2 sample responses of an ideal feedforward controller are compared to those of two optimal controllers. The latter are designed so that the mean square control effort is 88 and 67%, respectively, of that of the ideal controller. In the case of the ideal controller for this model, the manipulative variable called for by the controller is the mirror image of the disturbance. The control signal of the optimal controllers is attenuated, especially at input peaks. In cases where clipping occurs, far more saturation is obtained with the ideal controller than with the optimal controller.

The variation of the controlled output (lower channel) is shown in Figure 3 as a function of saturation level. Although all tests were carried out for a time period exceeding 250 system constants, the random nature of the disturbance caused considerable variation in the results. However, it can be seen that the most effective output attenuation was achieved by the ideal controller even under conditions where considerable clipping of the manipulative variable occurs. At very low saturation levels, the output from all three systems approaches the same value since all were saturated most of the time.

The discussion shows that the optimal controller does not perform its prime function—disturbance attenuation —better than the simple ideal controller of (5) even under conditions of constrained control effort. The principal improvement is that the optimal controller saturates the constrained control boundaries a smaller fraction of the time. Although control saturation may not be of importance for some applications, such behavior frequently implies other undesirable nonlinear effects such as hysteresis and backlash. These effects can cause further important deterioration of control efficiency. The responses referred to above where the control effort has been clipped cleanly may tend to underemphasize this beneficial effect of optimal control. Nonetheless, it is obvious that even if control saturation is to be avoided, a sanguine estimate of the amount of available control effort should be used for design purposes. An optimal controller will not utilize more control effort than its design value and therefore will sacrifice some control efficiency if greater control effort should be available.

OPTIMAL CONTROL WITH DEAD TIME

Other reasons exist besides control effort constraint

TABLE 1. VARIATION OF CONTROLLER TIME CONSTANTS AS FUNCTION OF DISTURBANCE FREQUENCY

$$\text{System: } C = \frac{D + M}{2 + s}$$

$$\text{Controller: } M = -\frac{T_o + T_1 s}{1 + \beta s} D$$

Frequency Radions/.unit time	Mean square output = .01 T_o	Mean square output = .01 T_1	Mean square output = .3 T_o	Mean square output = .3 T_1
0.1	.90	17.0	.55	3.5
0.4	.85	16.0	.53	3.5
1.0	.80	15.0	.50	3.4
4.0	.70	14.0	.30	3.3
10.0	.60	12.0	.20	3.2

which prevent realization of the perfect output attenuation by Equations (2), (4), and (5). One such reason is that a dead time may exist in control action. If, for example, a change in cooling water flow is required, mechanical, pneumatic, and process factors may cause a finite delay between the time when the control signal is generated and the time that the flow correction exerts influence on the process.

Controller dead time is represented as an exponential dealy factor $e^{-\tau_M s}$ in the controller transfer function P_M. This factor implies that the factor m (t) has a positive shift along the time axis. In the controller ratio $-P_D/P_M$ given by (5), a positive exponential factor results requiring the controller Q_D to shift its signal in a negative time direction. Physically the controller is being called upon to predict the value of the random disturbance before it occurs. Such a controller is, of course, impossible to construct, which means that perfect output attenuation is not possible. As in the previous case, no a priori reason exists implying that the best possible control performance will be realized by the ideal controller (5).

Since Wiener's optimization techniques also apply to systems containing dead times, optimal controllers were computed for the same model as considered in the previous section except that factors of the form $e^{-\tau_M s}$ were considered in the controller transfer function. That is

$$C = \frac{K_D}{\alpha + s} D + \frac{K_M}{\alpha + s} e^{-\tau_M s} M \qquad (14)$$

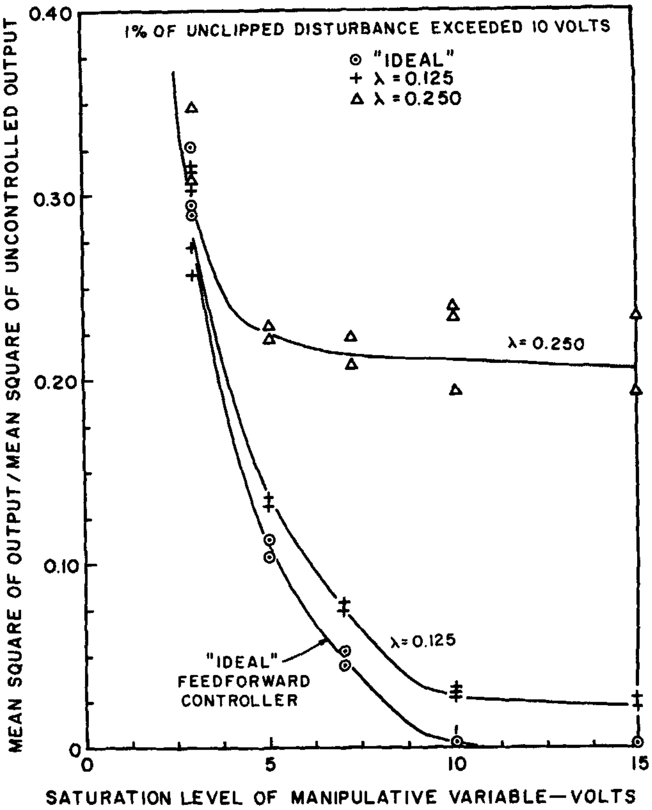

Fig. 3. Analog computer response results for various feedforward control systems as a function of controller saturation.

where as before $K_M = K_D = 1$, $\alpha = 2$, and τ_M takes on parametric positive values. The random disturbance described in (9) was again assumed with a mean frequency of 1.5. Since the time delay τ_M applies equally to both feedforward and feedback control, feedforward control without feedback is to be used again for this case.

The relationship between available control effort and output attenuation is shown in Figure 1 for the system of (14) with various values of the parameter τ_M (dead time). The uncontrolled response approaches the same finite limit as in the preceding case when the allowable control effort approaches zero. In the present situation there is not, however, a finite on the control effort required for perfect attenuation. The size of the dead time limits the degree of control that can be achieved regardless of the amount of control effort that is applied.

The optimal feedforward control function has one pole

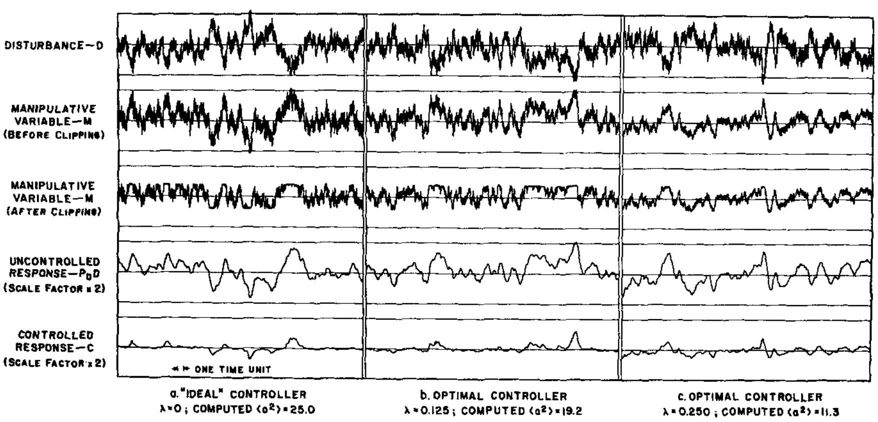

Fig. 2. Response of feedforward controlled system to a random disturbance with clipping at 50% of maximum disturbance level.

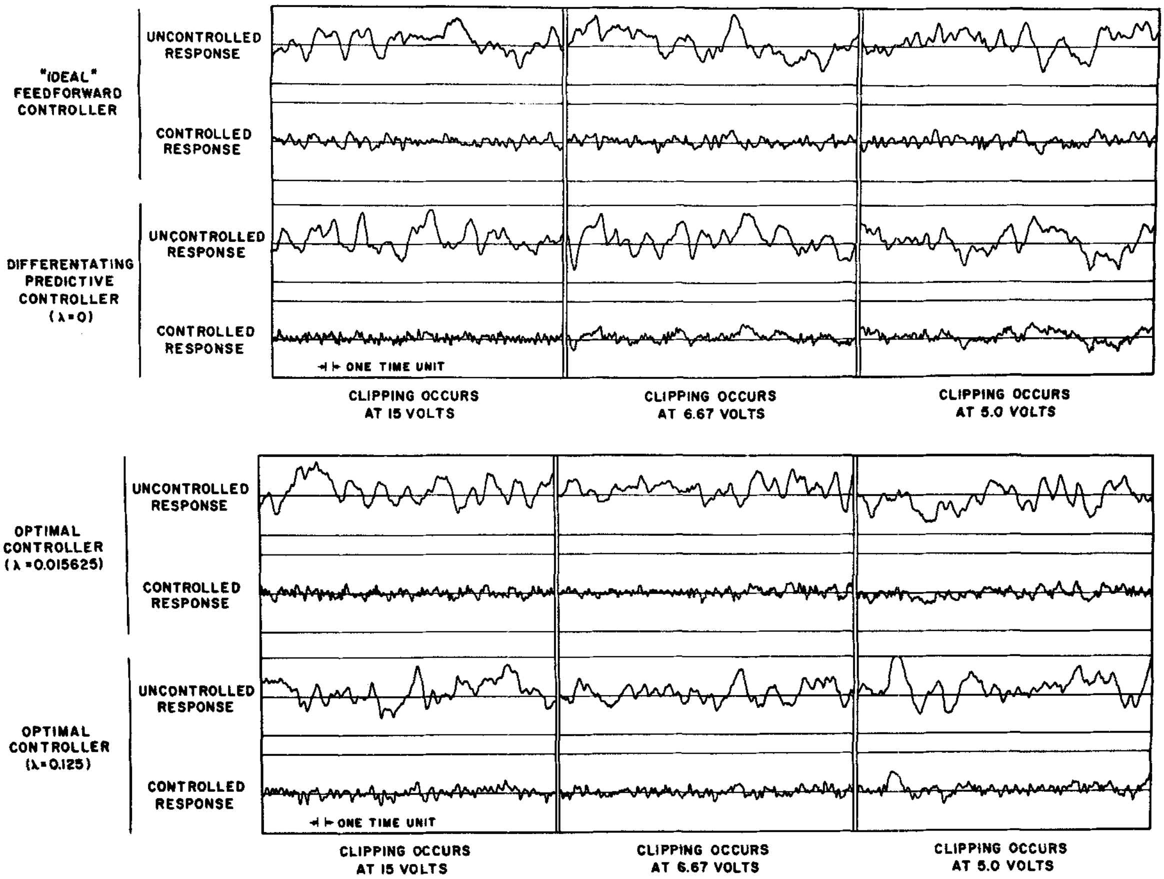

Fig. 4. Response of feedforward controlled system with dead time, τ_M, of 0.1 for a random disturbance.

and one zero as given by (13). When dead time was assumed absent and the constraint on control effort was relaxed, the time constants of the numerator and denominator both approached a common small value so that in the limit the controller approached the constant defined by (5) and (8). When dead time is assumed present, relaxation of the control effort constraint for the optimal controller results in the denominator time constant vanishing while the numerator time constant approaches a finite value. Thus the optimal controller for a system with dead time is a differentiator which bases control action on the trend or slope of the disturbance as well as on its instantaneous magnitude. Differentiation of the disturbance causes the mean square control effort to approach infinity in an attempt to achieve predictive control.

The responses shown in Figure 4 are analog computer results from a system described by (14) with a pure dead time τ_M of 0.10. The controller yielding the results in the top channel was the ideal controller that was so successful for control of the system previously considered. In this case, however, control was relatively poor.

The second set of curves results from the differentiator-predictor designed with a control effort weighting factor of zero, that is, where there is no constraint on control effort. The manipulative variable response has not been shown because the controller response for the differentiator was simply a series of high frequency fluctuations from one control limit to the other. This unconstrained optimal controller is easily the best at the highest level of control effort, but, as emphasized by Figure 5, it falls off to be the poorest when the ability of the controller to respond is restricted.

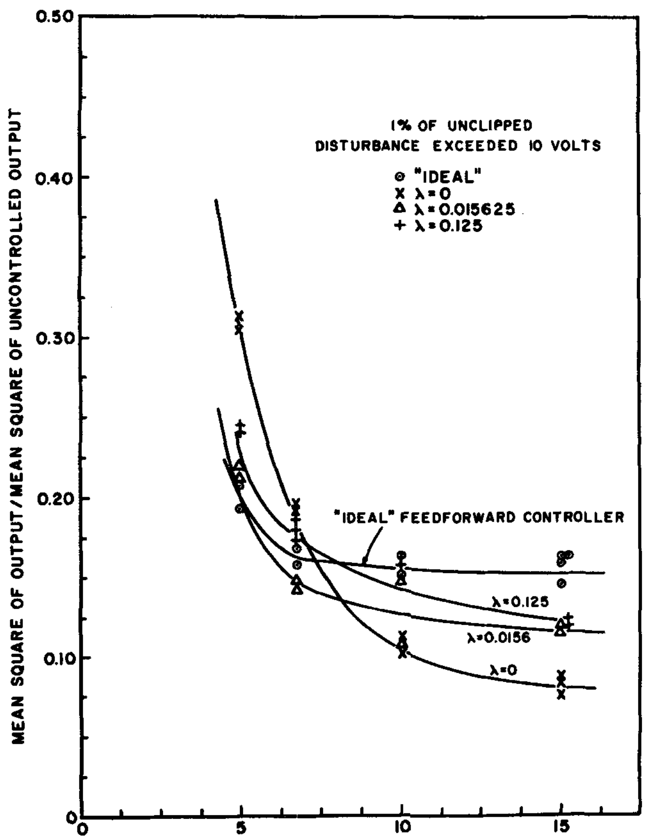

Fig. 5. Analog computer response results for various feedforward control systems with dead time, τ_M, of 0.125 as a function of *controller saturation.*

Table 2. Model Error Output of Optimal Controllers With and Without Feedback

Controls designed to minimize model error output				Controllers designed without consideration of model error	
Operating as designed		Operating with feedback removed			
$<\Delta c^2>$	$<c^2>$	$<\Delta c^2>$	$<c^2>$	$<\Delta c^2>$	$<c^2>$
.059	.2334	3.07	.222	3.03	.222
.060	.0451	3.92	.0338	3.62	.0338
.065	.0151	4.35	.0043	4.16	.0043
.063	.0071	4.47	.0004	4.33	.0004
.116	.8290	2.52	.823	2.76	.823
.119	.1152	3.38	.110	3.23	.110
.128	.0171	4.16	.012	3.93	.012
.125	.0067	4.47	.0004	4.33	.0004

The lower channels on Figure 4 show the controlled response of this system containing dead time where the controllers were optimized with constraints on control effort. The controllers correspond to those implied in the construction of the performance diagram, Figure 1. As indicated in Figures 4 and 5, efficiency of output attenuation and sensitivity to control effort clipping is a compromise between performance of the ideal controller and the unconstrained optimal controller. When sufficient control effort is available, a control design similar to the constrained optimal would seem reasonable. However, it is clear that a considerable reduction in performance must be expected in the control of a system containing dead time regardless of the design used.

OPTIMIZATION IN THE PRESENCE OF MODEL ERROR

Even in the absence of control effort constraints and controller dead time, perfect output attenuation is not achieved in practice because of model inaccuracies. Model error can be divided into two classes: (1) permanent error, that is, a value of a parameter which has been assumed or measured inaccurately; or (2) transient error, that is parameter values which change with time. The former type of error may succumb to development of accurate model identification techniques or can be tuned out by adaptive control practices. However the variability of transient error will cause persistent model error output. Examples of the latter range from scaling of heat transfer surfaces and ambient temperature cycling to variations of catalyst concentration or catalyst activities in systems where other variables are the measured forcing functions. System nonlinearities can also cause apparent variation in the linear parameters.

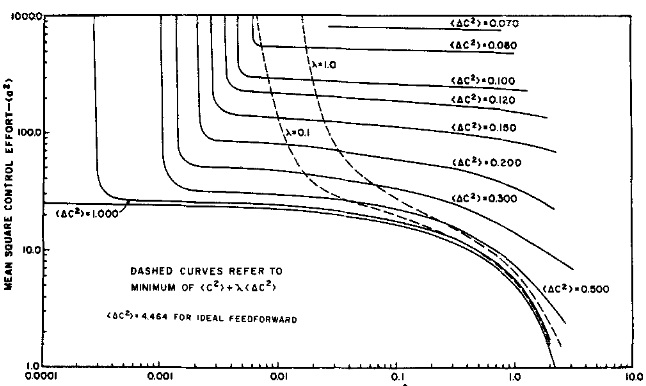

Fig. 6. Response characteristics for optimal composite control of first order system with parameters of model error output, $<\Delta c^2>$. Mean square value of feedback noise is 10^{-4}.

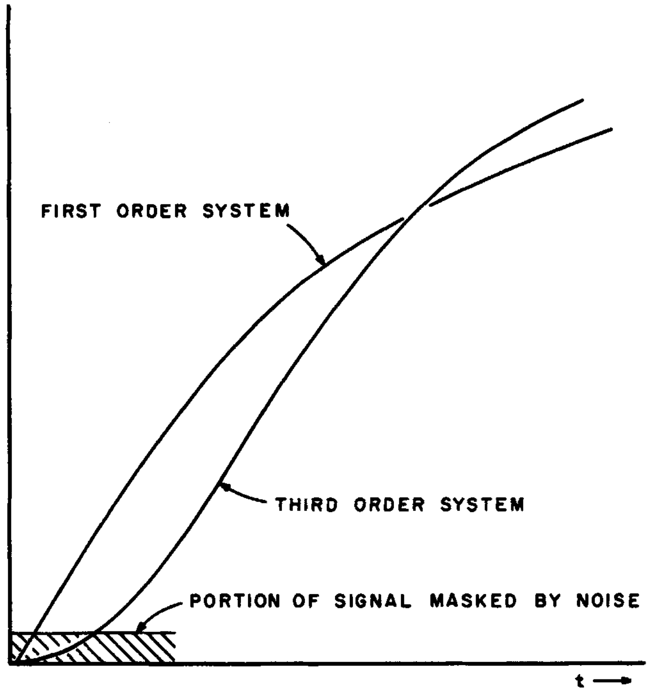

Fig. 7. System response to step functions.

In short, this type of error is a form of unmonitored disturbance which is regarded here in relation to the transfer functions of monitored variables. A truly optimal controller must take into consideration sensitivity to this type of error.

Computation of the model error factor will be made by determination of the incremental response due to incremental changes to the system parameters. Formally this is given as

$$\Delta C = \Delta P_D D + \Delta P_M M \qquad (15)$$

where the Δ's represent incremental changes.

The optimal control system is designed to minimize mean square output $<c^2>$ subject to the conditions

$$<\Delta c^2> \; < \mathcal{L}_1^2 \qquad (16)$$

and

$$<a^2> \; < \mathcal{L}_2^2 \qquad (17)$$

These conditions are satisfied by minimization of the sum

$$F = <c^2> + \lambda^2<\Delta^2> + \eta^2<a^2> \qquad (18)$$

The weighting factors in the last equation can be permuted between any of the variables without affecting the results.

If the sum in (18) is minimized for models without dead time, infinite feedback gain results for any system with finite model error (*11*). Infinite feedback gain does *not* necessarily imply infinite control effort; it merely implies a feedback controller which is equivalent to the ideal feedforward controller of (5). The feedforward portion of this optimal composite control merely modifies the ideal response sufficiently to meet control effort constraints.

If dead time is present, a feedback controller with finite gain will result. However even for systems without dead time infinite gain feedback is impractical. Noise and error in the feedback system are amplified along with the low level output signal. Modifications were made to (3) to take this factor into account by allowing the feedback controller to operate on a feedback noise δ, as well as on the output signal, that is

$$M = Q_D D - Q_C (c + \delta) \qquad (19)$$

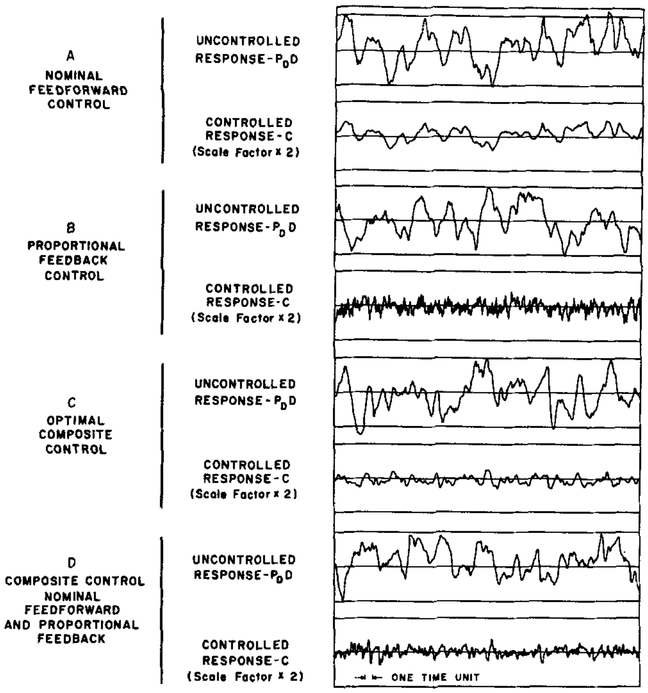

Fig. 8. Control of a first order system with model error and feedback noise. Error in plant gain = 0.25. Noise in feedback ≈ 8% of uncontrolled output.

The output-control effort diagram for a system with model error is shown in Figure 6. This diagram was computed for mutually independent mean square errors of 0.25 in each of the model parameters of a first order system similar to (14) and (19). The mean value of all dead times was assumed to be zero in order to focus attention on the system response to model error. An input was assumed from dead time error, but this could be considered equally well as the result of an unmonitored disturbance.

Lines of constant error output or sensitivity (*19*) are nearly horizontal over a wide range of response indicating that, as in the case for errorless systems, nominal output at a constant sensitivity is a weak function of control effort. The abrupt rise of control effort for a given sensitivity at low levels of nominal output represents the system response introduced by feedback amplification of the noise factor in (19).

Despite the coupled optimization technique, sensitivity reduction was achieved almost entirely by the use of feedback. In Table 2 the error output for several optimal controllers is compared to the error output of the same controllers when feedback is removed. The error output of the optimal controllers without feedback increases to approximately the same level as that of a controller designed for an errorless model. Optimization of other models in which the effectiveness of feedback was restricted, such as by a high noise level or significant dead times in the feedback system, showed that important sensitivity reduction cannot be achieved by feedforward control alone. However, other calculations for models with controller dead times yielded imperfect differentiating controllers which were far more sensitive to model error than controllers for no dead time system. Thus it is seen that the need to predict control action yields optimal controllers which are very sensitive to both the available effort and the model parameters used as the design basis.

The presence of error is more serious in some plant

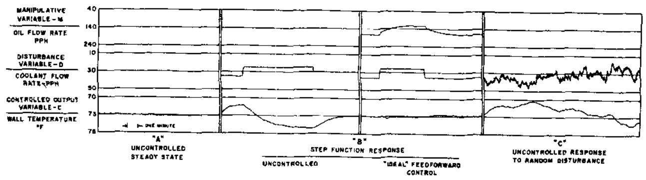

Fig. 9. Response of experimental system to disturbances.

parameters than in others. A comparison of model error output for the same relative error in each of the plant parameters is given in Table 3. A mean square model error of 0.25 was assumed separately in each of the elements of the following model:

$$C = \frac{K_D(1 + R_D s)e^{-\tau_c s}\,D + K_M(1 + R_M s)e^{-(\tau_c+\tau_M)s}\,M}{(\alpha + s)\;(1 + R_1 s)} \qquad (20)$$

where the mean value of the parameters are

$$\langle K_D\rangle = \langle K_M\rangle = 1,$$
$$\langle R_D\rangle = \langle R_M\rangle = \langle R_1\rangle = 0.5,$$
$$\langle\alpha\rangle = 2,$$
$$\langle\tau_c\rangle = \langle\tau_M\rangle = 0.5$$

The model error output was computed on the basis of ideal feedforward control without feedback for a disturbance with a mean frequency of 1.5 and a mean square amplitude of 25.

Model error output was zero for variations in pole locations and in feedback dead time since feedforward control does not depend on system natural frequency or output sensing. However, when these errors are present along with others, interactions occur which do produce model error output especially in the design of the feedback portion of the control.

The smallest feedforward model error output resulted from error in the time constants of the plant zeros. Only slightly more error output resulted from gain inaccuracies. The largest effect resulted from error in controller dead time. As mentioned earlier, the model error output would be considerably greater if an optimal partial differentiator were designed instead of the ideal controller.

In this investigation up to third order systems were considered. However, since the optimization technique is dependent essentially on cancellation of plant poles and substitution of controller poles with the desired characteristics, the results for the first order systems extend directly to high order models with one important exception. If the net system order is higher, feedback noise and dead time exert greater influence on optimal

TABLE 3. MODEL ERROR OUTPUT CAUSED BY PARAMETER VARIATION OF 0.25 FOR INDIVIDUAL ELEMENTS OF MODEL

$$C = \frac{K_D\,(1 + R_D s)e^{-\tau_C s}\;D + K_M\,(1 + R_M s)e^{-\tau_M s}\;M}{(\alpha + s)\;(1 + R_1 s)}$$

Term containing error	Mean square model error output, $\langle\Delta c^2\rangle$
K_D	0.8929
K_M	0.8929
R_D	0.7653
R_M	0.7653
α	0
R_1	0
τ_M	2.6784
τ_C	0

TABLE 4. EFFECTIVENESS OF CONTROLLERS FOR EXPERIMENTAL SYSTEM

Type of controller	Controller efficiency $1 - \dfrac{\dfrac{\int\lvert c\rvert dt}{\int\lvert d\rvert dt}\text{ controlled}}{\dfrac{\int\lvert c\rvert dt}{\int\lvert d\rvert dt}\text{ uncontrolled}}$	Relative control effort $\dfrac{\dfrac{\int\lvert m\rvert dt}{\int\lvert d\rvert dt}\text{ controlled}}{\dfrac{\int\lvert m\rvert dt}{\int\lvert d\rvert dt}\text{ optimal C}}$
Random disturbance:		
Optimal A	0.922	0.89
Optimal B	0.815	0.72
Optimal C	0.963	1.00
Ideal feedforward	0.916	0.75
Proportional feedback	0.843	0.94
Ideal F.F. plus proportional F.B.	0.973	1.06
Step input:		
Ideal feedforward	0.926	—

control sensitivity. An explanation for this lies in consideration of the third order system response to a step function. (Figure 7). The s-shaped response curve starts very gradually so that the output is masked by noise for a longer period of time than for first order response, thus preventing effective feedback control.

Analog computer results for control of a first order system with model error and feedback noise is illustrated in Figure 8. Performance of the optimal composite controller is clearly superior to control by either feedback or feedforward control alone. However, a composite controller consisting of ideal feedforward and proportional feedback control produces an output of comparable quality. As indicated by performance diagrams, the optimal control achieved only a small conservation of control effort.

EXPERIMENTAL RESULTS

In order to examine the validity of previous digital and analog computations, optimal controls were applied to a physical system. The process to be controlled is a perfectly stirred, jacketed vessel with cold and hot fluids entering the jacket and stirred center, respectively. The control objective is to vary the flow rate of the hot fluid so that the temperature of the separating wall remains constant despite variations in flow rate of the cold fluid.

The mathematical model for this system was determined experimentally by a time domain identification technique developed by Heymann (7) to be

$$C = \frac{(1 + 1.33s)e^{-0.35s}D + (1 + 0.33s)e^{-0.7s}M}{(1 + 0.60s)(1 + 1.38s)} \quad (21)$$

The experimental results indicated in Figure 9 show that an accurate model identification was achieved; ideal feedforward control based on this model achieved an attenuation of 93% of the uncontrolled response to square wave input.

Three optimal controllers were designed for this system: A, consisting primarily of feedforward for output attenuation; B, consisting primarily of feedback for resistance to model error; and C, a compromise using greater control effort to achieve reduction of both quantities.

Comparison of the controlled response shown in Figure 10 with the typical uncontrolled response shown in Figure 9c illustrates the high degree of control achieved by all three of the optimal controllers. The control as achieved by the composite control C surpassed the performance of A because the feedback noise in this system was quite low.

As was found in previous analog computer tests, approximately equivalent performance was achieved by use of nominal controllers as shown in Figure 11. The absolute value of the disturbance, output, and manipulative variable were integrated to compute the efficiencies shown in Table 4. As can be seen, the optimal controllers used somewhat less control effort than the equivalent nominal control but it would seem difficult to base a design on this differenec.

SUMMARY AND CONCLUSIONS

The design equations including all of the constraints yield physically realizable controls which give real performance that corresponds broadly to the intuitive ideas leading to their development. In the absence of significant model error and dead times, primarily feedforward controllers are specified. If the system is assumed to be constrained by available control effort magnitude, optimal control laws result which attenuate the output somewhat more poorly than an ideal feedforward controller even under conditions of constrained control effort. The optimal control, however, saturates a smaller fraction of the time so that superior optimal output attenuation would result if saturation were accompanied by other undesirable effects such as hysteresis. It would be recommended that in most cases that implementation of optimal controls should be based on optimistic estimates of available control effort since the optimal controller will not use more control effort for output attenuation even if it is available. If control saturation does not invoke extra penalties, and in the absence of control dead time, ideal feedforward is superior to optimal feedforward.

For systems containing dead times, significant improvements in control performance can be achieved with the optimal design. However, this controller is a differentia-

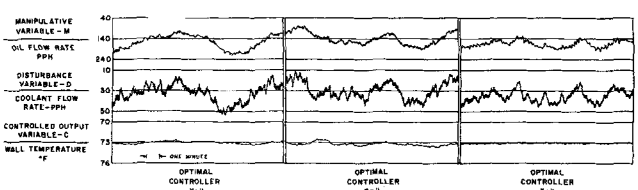

Fig. 10. Response of experimental system with optimal composite controllers from text.

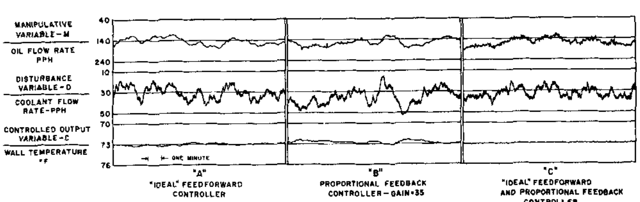

Fig. 11. Response of experimental system with nominal controllers.

tor which is more sensitive than an ideal controller, not only to constraints of control effort but also to error in the mathematical model. Optimal design for these cases should be made with less optimistic estimates of available control effort.

When one includes consideration of model error in the design equations, feedback control is specified in addition to feedforward. In fact, attenuation of output resulting from model error can be affected only slightly by modification of the feedforward controller and must therefore be achieved almost entirely by feedback. If noise associated with the feedback system is low and if dead times are absent from the output sensing and amplification circuits, the optimal controller tends to consist primarily of feedback. There is a definite limit, however, to the degree of output attenuation with the use of feedback. Amplification of noise and sensing error produce more output than is being eliminated when extremely low levels of output are sought. Furthermore, small dead times in real systems initiate oscillations when large feedback gains are employed. These same factors limit the degree of sensitivity reduction that can be achieved by feedback.

For the carefully stabilized and identified experimental system, we found that either feedforward or feedback control yielded very good results. However, the marked superiority of the composite control was not only indicated from the computations, but was also clearly demonstrated by the experimental study. When the very best control is desired, this form is indicated.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant GK 98. One of the authors (R. H. Luecke) was the recipient of a National Science Foundation Cooperative Fellowship during a major portion of this work. The authors are indebeted to the staff of the Osage Computer at the University of Oklahoma for their generous contribution of computer time and programming assistance.

NOTATION

$A, A(s)$ = Laplace transform of $a(t)$
a_i = constant
b_j = constant in system describing equation
$C, C(s)$ = Laplace transform of $c(t)$
$c(t)$ = output or controlled variable
$D, D(s)$ = Laplace transform of $d(t)$
$d(t)$ = disturbance or load variable
$F(A, \lambda)$ = sum of integrals to be minimized
g_i = constant in system describing equation
K_D = gain factor for disturbance transfer function
K_M = gain factor for manipulative variable transfer function
$\mathcal{L}$ = maximum value of mean square control effort
$\mathcal{L}_1$ = constraint on model error output
$\mathcal{L}_2$ = constraint on control effort
$M, M(s)$ = Laplace transform of $m(t)$
$m(t)$ = manipulative or control variable
$P_A, P_A(s)$ = linear operator on manipulative variable to determine constraining conditions
$P_D(s)$ = transfer function of disturbance in system Laplace domain equation
$P_M(s)$ = transfer function of manipulative variable in system Laplace domain equation
$Q_C(s)$ = transfer function of feedback controller
$Q_D(s)$ = transfer function of feedforward controller
R_D = time constant of zero of disturbance transfer function
R_M = time constant of zero of manipulative variable transfer function
R_1, R_2 = time constants of poles of plant transfer function
s = Laplace transform variable
T_1, T_2 = control transfer functions

Greek Letters

α = system natural frequency or pole of transfer function
β = factor defined
ΔC = model error output or increment in output due to parameter variation
ΔP_D = error in disturbance transfer function
ΔP_M = error in manipulative variable transfer function
δ = random noise in feedback circuit
η = weighting factor in Equation (18)
$\lambda, \lambda_1, \lambda_2$ = Lagrange multiplier or weighting factors
μ = magnitude factor of disturbance
σ = mean frequency of disturbance
τ_C = dead time in output circuit
τ_M = dead time in controller
$\Phi_{AB}(s)$ = cross spectral density of random time functions, $a(t)$ and $b(t)$
$<\ldots>$ = mean value

LITERATURE CITED

1. Bollinger, R. E., and D. E. Lamb, "1963 Joint Automatic Control Conference," Preprint XVIII-4, Univ. Minn., Minneapolis (1963).
2. Buckley, P., "Techniques of Process Control," Wiley, New York (1964).
3. Calvert, S., and G. Coulman, *Chem. Eng. Progr.*, **57**, 9, 45 (1961).
4. Chang, S. S. L., "Synthesis of Optimum Control Systems," McGraw-Hill, New York (1961).
5. Harris, J. T., and R. S. Schechter, *Ind. Eng. Chem. Fundamentals*, **2**, 3, 245 (1963).
6. Haskins, D. E., and C. M. Sliepcevich, *ibid.*, **4**, 3, 241 (1965).
7. Heymann, M., Ph.D. thesis, Univ. Oklahoma, Norman (1965).
8. Horowitz, I. M., *IRE Trans. Auto. Control*, AC-5, p. 5 (1959).
9. Kipiniak, W., "Dynamic Optimization and Control, A Variational Approach," Mass. Inst. Technol. Press and Wiley, New York (1961).
10. Laning, J. H., and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill, New York (1956).
11. Luecke, Richard H., Ph.D. thesis, Univ. Oklahoma, Norman (1966).
12. Lupfer, D. E., and J. R. Parsons, *Chem. Eng. Progr.*, **5**, 9, 37 (1962).
13. Luyben, W. L., and J. A. Gerster, *Ind. Eng. Chem. Fundamentals*, 3, 4, 106 (1964).
14. MacMullan, E. C., and F. G. Shinskey, *Control Engineering*, (1964).
15. Merriam, C. W., "Optimization Theory and the Design of Feedback Control System," McGraw-Hill, New York (1964).
16. Newton, G. C., L. A. Gould, and J. F. Kaiser, "Analytical Design of Linear Feedback Controls," Wiley, New York (1957).
17. Peterson, E. L., "Statistical Analysis and Optimization of Systems," Wiley, New York (1961).
18. Solodovnikov, V. V., "Introduction to the Statistical Dynamics of Automatic Control Systems," Cover Publications, Inc., New York, (1960).
19. Tomovic, R., "Sensitivity Analysis of Dynamic Systems," McGraw-Hill, New York (1963).
20. Wiener, N., "Extrapolation, Interpolation and Smoothing of Stationary Time Series," Mass. Inst. Technol. Press and Wiley, New York (1949).

Manuscript received September 27, 1966; revision received November 17, 1966; paper accepted May 17, 1967. Paper presented at AIChE Houston meeting.